In the complex plane $(O, \vec{u}, \vec{v})$, consider the points M, M' and A of respective affixes z, z' and i. Given the relation $z' = \dfrac{iz+1}{z}$ where $z \neq 0$

1. Find the algebraic form of z' when $z = \dfrac{2-i}{5}$
2. Find the exponential form of z when $z' = 1$
3. Suppose that $z = x + iy$ and $z' = x' + iy'$ where x, x', y and y' are real numbers.

   a. Show that $x' = \dfrac{x}{x^2+y^2}$ and $y' = \dfrac{x^2+y^2-y}{x^2+y^2}$

   b. Find the locus of M when z' is pure imaginary.

   c. Verify that $z' = \dfrac{i(z-i)}{z}$

   d. Deduce that $OM' = \dfrac{AM}{OM}$ and $\boxed{\arg(z') = \dfrac{\pi}{2} + \left(\vec{OM}, \vec{AM}\right) + 2k\pi \, . \, (k \in Z)}$

   e. Find the locus of the point M as M' moves on the circle with center O and radius 1.

   f. If the triangle OAM is direct equilateral, write the trigonometric form of z'.

Exercise:

The complex plane is referred to a direct orthonormal system $(o, \vec{u}, \vec{v})$

Consider the points A, B, C, and D such that $z_A = i$, $z_B = -2i$, $z_C = \frac{1}{2}i$ and $z_D = 1$

E is the point so that the triangle ODE is isosceles with vertex D, and $(\vec{u}, \overrightarrow{DE}) = \frac{\pi}{6}$

1. Write $\frac{z_{\overrightarrow{DE}}}{z_{\overrightarrow{OD}}}$ in exponential form then in algebraic form.
2. Deduce that $Z_E = (1+\frac{\sqrt{3}}{2}) + \frac{1}{2}i$ and $|Z_E| = \sqrt{2+\sqrt{3}}$
3. Use the triangle ODE to show that $\arg(Z_E) = \frac{\pi}{12}$ then use part (2) to calculate the exact values of $\cos(\frac{\pi}{12})$ and $\sin(\frac{\pi}{12})$
4. z is the affix of a variable point M and z' is that of a point M' with $z' = \frac{2z-i}{iz+1}$

   a. If $z = e^{i\frac{\pi}{6}}$, write z' in exponential form.
   b. If z' is pure imaginary, prove that z is also pure imaginary.
   c. Prove that $(z'+2i)(z-i) = 1$.
   d. Deduce the value of BM'×AMand the measure of $(\vec{u}, \overrightarrow{BM'}) + (\vec{u}, \overrightarrow{AM})$